Pursuit Evasion Games with Imperfect Information Revisited

Barak Or[1], Joseph Z. Ben-Asher[2] and Isaac Yaesh[3]

Abstract

Differential Games for pursuit evasion problems have been investigated for many years. Differential games, with linear state equations and quadratic cost functions, are called Linear Quadratic Differential Game (LQDG). In these games, one defines two players a pursuer and an evader, where the former aims to minimize and the latter aims to maximize the same cost function (zero-sum games). The main advantage in using the LQDG formulation is that one gets Proportional Navigation (PN) like solutions with continuous control functions. One approach which plays a main role in the LQDG literature is Disturbance Attenuation (DA), whereby target maneuvers and measurement error are considered as external disturbances. In this approach, a general representation of the input-output relationship between disturbances and output performance measure is the DA function (or ratio). This function is bounded by the control. This work revisits and elaborates upon this approach. We introduce the equivalence between two main implementations of the DA control. We then study a representative case, a "Simple Boat Guidance Problem" (SBGP), with perfect and imperfect information patterns. By the derivation of the analytical solution for this game, and by running some numerical simulations, we develop the optimal solution based on the critical values of the DA ratio. Finally, we study a representative Missile Guidance Engagement (MGE) problem. The qualitative and quantitative properties of the MGE solution, based on the critical DA ratio, are studied by extensive numerical simulations, and are shown to be different than the fixed DA ratio solutions.

1. Introduction

Pursuit-evasion differential games have attracted a considerable attention since the seminal works of [1] and [2]. A class of differential games, with a couple of players driving linear state equations both affecting a quadratic cost function, are called Linear Quadratic Differential Game (LQDG). In these games, where the pursuer tries to minimize a quadratic cost function, whereas the evader tries to maximize the same cost function (i.e., these are zero-sum games). The cost function includes weights the squared miss-distance, the control efforts of both players, as well as occasionally trajectory shaping terms. The main LQDG formulation allows derivation of popular guidance laws such as Proportional Navigation (PN), Optimal Rendezvous (OR) and so on. A problem which is closely related to the LQDG problem is the one of Disturbance Attenuation

[1] Graduate Student, Faculty of Aerospace Engineering, Technion, Haifa
[2] Professor, Faculty of Aerospace Engineering, Technion, Haifa
[3] IMI Systems, Control Department

(DA), where pursuer actions are considered to be control actions, whereas all external actions such as target maneuvers and measurement errors, are considered to be disturbances. In fact, the DA problem is just one side of the double inequality forming the saddle-point relation and leads to the H^{∞} problem. DA problems can be either deal with perfect information patterns where both pursuer and evader share a perfect information regarding the full state-vector and their both actions or imperfect information where both players have access only to noisy measurements of a linear combination or part of the state-vector. The present paper, revisits DA problems in the latter case, and provides detailed analysis of simple pursuit-evasion examples which allow insight to the interplay between the control and estimation parts of the pursuer strategy.

We firstly introduce the equivalence between two main implementations of the DA control, one formulated by Speyer [1] and the other by Limbeer [2]. Secondly, we introduce and discuss a representative case study of a Simple Boat Guidance Problem (SBGP), with perfect and imperfect information patterns. We derive the optimal solution, and present numerical results, for the SBGP using critical and non-critical values of the DA. We then introduce another representative case study, concerning Missile Guidance Engagement (MGE). The qualitative and quantitative properties of the MGE solution, based on the critical and non-critical DA values, were studied by extensive numerical simulations.

One main interesting result can be found in [5]-[6]. A case study for MGE with jamming power was introduced [Figure 1]. The Navigation Constant (connecting line-of-sight rate with the acceleration's command) grows with the jamming magnitude (in fact, it grows when the cost imposed on the noise in the game is reduced). This result is somewhat counter-intuitive. For example, in another case study [7] Navigation Constant gets lower as the disturbances grows. The present paper clarifies this result as well.

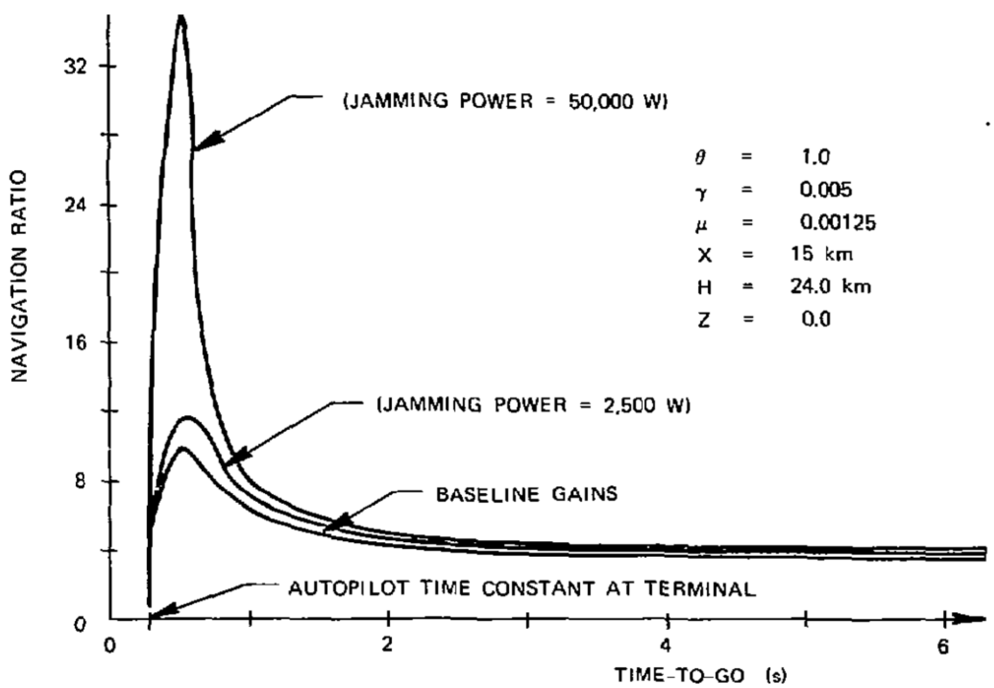

Figure 1- Navigation Constant vs. Jamming power [J.L Speyer]

2. The Equivalence between Two Implementations of the Pursuer Strategy

Consider the following system as can be found also in [3], [8]:

$$\dot{x} = Ax + B_1 w + B_2 u \qquad , x(0) = x_0 \tag{2.1}$$

$$y = C_2 x + D_{21} w \tag{2.2}$$

$$z = C_1 x + D_{12} u \tag{2.3}$$

Where $x \in \mathbb{R}^n$ is the state vector, $w \in \mathbb{R}^q$ is an exogenous disturbance, $u \in \mathbb{R}^s$ is the control input signal, x_0 is an unknown initial state, $y \in \mathbb{R}^r$ is the measurement output. The matrices $A, B_1, B_2, C_1, C_2 . D_{12}, D_{21}$ are constant matrices of the appropriate dimensions.

$$D_{21}{}^T B_2 = 0 \qquad D_{21} D_{21}{}^T = I \qquad D_{12}{}^T C_1 = 0 \qquad D_{12} D_{12}{}^T = I \tag{2.4}$$

Consider also the following cost function

$$J = x^T(t_f) X_f x(t_f) - x_0{}^T Y_0^{-1} x_0 + \int_0^{t_f} \left(z^T z - \gamma^2 w^T w \right) dt \tag{2.5}$$

Which is to be respectively minimized and maximized by u and w. In connection with the Linear Quadratic Differential Game (LQDG) problem, the following couple of modified Differential Riccati Equations (DRE) plays important role:

$$-\dot{X} = \hat{A}_2^T X + XA + C_1{}^T C_1 \tag{2.6}$$

$$\dot{Y} = \hat{A}_1 Y + YA^T + B_1 B_1{}^T \tag{2.7}$$

Where:

$$\hat{A}_1 \triangleq A + \gamma^{-2} Y C_1{}^T C_1 - Y C_2{}^T C_2 \tag{2.8}$$

$$\hat{A}_2 \triangleq A + \gamma^{-2} S B_1 B_1{}^T X - B_2 B_2{}^T X \tag{2.9}$$

Also a third DRE, the solution of which Z is known to be related to those two, by

$$Z = \left(I - \gamma^{-2} YX \right)^{-1} Y \tag{2.10}$$

The LQDG literature presents the following solution to the above measurement feedback problem where w can take any full information strategy (i.e. with access to x) where u has access only to noisy measurement y.

The following solution has been found in [3]:

$$\dot{\hat{x}}_1 = \bar{A}_1\hat{x}_1 + \bar{B}_1 y \tag{2.11}$$

$$u = \bar{C}_1\hat{x}_1 \tag{2.12}$$

Where:

$$\bar{A}_1 = \hat{A} - B_2 B_2^T X\left(I - \gamma^{-2}YX\right)^{-1} \tag{2.13}$$

$$\bar{B}_1 = YC_2^T \tag{2.14}$$

$$\bar{C}_1 = -B_2^T X\left(I - \gamma^{-2}YX\right)^{-1} \tag{2.15}$$

Another solution appears in [4] which generally serves the control community:

$$\dot{\hat{x}}_2 = \bar{A}_2\hat{x}_2 + \bar{B}_2 y \tag{2.16}$$

$$u = \bar{C}_2\hat{x}_2 \tag{2.17}$$

Where:

$$\bar{A}_2 = \hat{A}_2 - ZC_2^T C_2 \tag{2.18}$$

$$\bar{B}_2 = ZC_2^T \tag{2.19}$$

$$\bar{C}_2 = -B_2^T X \tag{2.20}$$

In [3] the two solutions have been shown to coincide in the steady-state time-invariant case. Our aim is to show that, as could be expected, the solutions are equivalent. We use here a somewhat different approach which applies a similarity transformation between those two implementations.

To this end, consider $T > 0$, so that:

$$\hat{x}_2 = T^{-1}\hat{x}_1 \tag{2.21}$$

We readily obtain using the following identity, (2.22), that the two implementations are equivalent.

$$\frac{dT^{-1}}{dt} \equiv -T^{-1}\dot{T}T^{-1} \tag{2.22}$$

Therefore, $\exists T > 0$ s.t:

$$\bar{A}_1 = T\bar{A}_2 T^{-1} + \dot{T}T^{-1} \tag{2.23}$$

$$\bar{B}_1 = T\bar{B}_2 \tag{2.24}$$

$$\bar{C}_1 = \bar{C}_2 T^{-1} \tag{2.25}$$

We next intend to show that $T = I - \gamma^{-2}YX$ satisfies the above relation. Indeed,

$$\bar{C}_1 = -B_2^{\,T} X\left(I - \gamma^{-2}YX\right)^{-1} = \bar{C}_2 T^{-1} \tag{2.26}$$

Similarly,

$$\bar{B}_1 = YC_2^{\,T} = TZC_2^{\,T} = \left(I - \gamma^{-2}YX\right)ZC_2^{\,T} = T\bar{B}_2 \tag{2.27}$$

It, therefore, remains to establish the relation between $\bar{A}_1$ and $\bar{A}_2$. To this end, note that using the above definition of T we have:

$$\bar{A}_1 = \hat{A}_1 - B_2 B_2^{\,T} XT^{-1} \tag{2.28}$$

$$\bar{A}_2 = \hat{A}_2 - ZC_2^{\,T} C_2 \tag{2.29}$$

From (2.23) we have to show that:

$$\bar{A}_1 T = T\bar{A}_2 + \dot{T} \tag{2.30}$$

In other words,

$$\hat{A}_1 T - B_2 B_2^{\,T} X = T\hat{A}_2 - YC_2^{\,T} C_2 + \dot{T} \tag{2.31}$$

Where we have used the relation $TZ = Y$.

The expression for transformation derivative is given by:

$$\dot{T} = -\gamma^{-2}\dot{Y}X - \gamma^{-2}Y\dot{X} \tag{2.32}$$

Substitute (2.32) in (2.31) and using the DRE's of $(2.6),(2.7)$ yield the following:

$$L \triangleq \hat{A}_1 - \gamma^{-2}\left(\dot{Y} - B_1 B_1^{\,T} - YA^T\right)X - B_2 B_2^{\,T} X \tag{2.33}$$

$$R \triangleq \hat{A}_2 - \gamma^{-2}Y\left(-\dot{X} - C_1^{\,T} C_1 - A^T X\right) - YC_2^{\,T} C_2 - \gamma^{-2}\dot{Y}X - \gamma^{-2}Y\dot{X} \tag{2.34}$$

Finally, substitute $\hat{A}_1$, $\hat{A}_2$ and collecting terms, we readily find that $L = R$ as required, completing the proof of similarity based equivalence.

Note: One may choose the initial condition, and use a trivial transformation between the two implementations, in order to achieve this equivalency.

As a result, we can see easily the equivalence between the two controls:

$$u_1 = \bar{C}_1 \hat{x}_1 = -B_2{}^T X \left(I - \gamma^{-2} Y X \right)^{-1} \hat{x}_1 \quad (2.39)$$

$$u_2 = \bar{C}_2 \hat{x}_2 = -B_2{}^T X \hat{x}_2 \quad (2.40)$$

$$\hat{x}_2 = T^{-1} \hat{x}_1 \quad (2.41)$$

$$T = I - \gamma^{-2} Y X \quad (2.42)$$

$$u_2 = -B_2{}^T X T^{-1} \hat{x}_1 = -B_2{}^T X \left(I - \gamma^{-2} Y X \right)^{-1} \hat{x}_1 \quad (2.43)$$

$$u_1 \equiv u_2 \quad (2.44)$$

■

3. Special Case-The Simple Boat Guidance Problem (SBGP)

Consider two boats, A and B, where A (the pursuer) wants to hit boat B (the evader). In order to accomplish this mission, boat A controls it's heading angle α , trying to navigate toward boat B. On the other hand, boat B tries to evade from boat A, and does that by controlling the heading angle β .

Assumptions:

- Two dimensional problem.
- The boats have constant velocities: V_A and V_B .
- Boat A is faster than Boat B: $V_A > V_B$.
- The boats have a direct control over their heading angles α and β , also $\alpha \ll 1, \beta \ll 1$

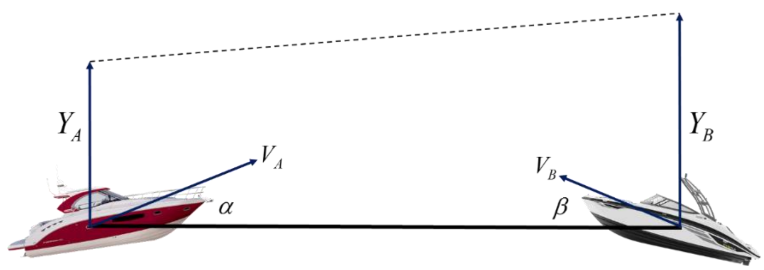

Figure 2-The SBGP Geometry Description

The equations of motion are:

$$x \triangleq Y_B - Y_A \qquad x(0) = 0 \tag{3.1}$$

$$\dot{x} = V_B \sin\beta - V_A \sin\alpha \triangleq w + u \tag{3.2}$$

$$z = x + v \tag{3.3}$$

Where $x \in \mathbb{R}^1$ is the "relative separation", $z \in \mathbb{R}^1$ is the measurement and $v \in \mathbb{R}^1$ is the additive noise. We can formulate the problem as "Min-Max problem", where boat A aims to minimize the relative separation at the terminal time, whereas boat B wants to maximize it.

3.1 Optimal Solution- Perfect Information Game

Consider first the perfect information game- where there is no measurement noise. From (3.2) and (2.6) we get:

$$\dot{x} = w + u \tag{3.4}$$

$$\min_u \max_w J(u,w) = \frac{b}{2}x^2(t_f) + \frac{1}{2}\int_0^{t_f}\left(u^2(t) - \gamma^2 w^2(t)\right)dt \tag{3.5}$$

$$\dot{X}(t) = \left(1-\gamma^{-2}\right)X^2(t) \quad X(t_f) = b \tag{3.6}$$

The optimal strategies are given by [10]:

$$u^*(t) = -X(t)x(t) \tag{3.7}$$

$$w^*(t) = \gamma^{-2}X(t)x(t) \tag{3.8}$$

$$X(t) = \frac{1}{\left(1-\gamma^{-2}\right)\left(t_f - t\right) + 1/b} \tag{3.9}$$

Where $\gamma > 1$ guarantees a Positive Definite $X(t)\ \forall t$;

One gets the linear control feedbacks (player's strategies):

$$u^*(t) = -\frac{1}{\left(1-\gamma^{-2}\right)\left(t_f - t\right) + 1/b}x(t) \tag{3.10}$$

$$w^*(t) = \gamma^{-2}\frac{1}{\left(1-\gamma^{-2}\right)\left(t_f - t\right) + 1/b}x(t) \tag{3.11}$$

Note that by letting $b \to \infty, \gamma \to \infty$, we get a simple Collision Course Guidance (CCG) Law:

$$u^*(t) = -\frac{1}{\left(t_f - t\right)}x(t) \tag{3.12}$$

$$w^*(t) = 0 \tag{3.13}$$

3.2 Optimal Solution- Imperfect Information Game

Consider imperfect information such that there is noise, v, added to our measurements. From (2.5) we get

$$\max_{w}\min_{u} J = -\frac{1}{2}\gamma^2 Y_0^{-1} x^2(t_0) + \frac{1}{2} b x^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f}\left(u^2 - \gamma^2\left(w^2 + V^{-1}v^2\right)\right)dt \tag{3.14}$$

$$\dot{x} = u + w \tag{3.15}$$

$$z = x + v \tag{3.16}$$

Notice that we have added a weight V^{-1} to the quadratic term of v (in Eq. (2.5) the noise was normalized by V^{-1}). From (2.12) and (2.15), the pursuer optimal control is given by:

$$u^* = -\frac{X}{1-\gamma^{-2}YX}\hat{x} \tag{3.17}$$

$$\Lambda\left(t,\gamma^2,Y,X\right) \triangleq -\frac{X}{1-\gamma^{-2}YX} = -\frac{1}{X^{-1}-\gamma^{-2}Y} \tag{3.18}$$

Where:

$$X\left(t_f\right) = b \tag{3.19}$$

$$\dot{X} = X^2\left(1-\gamma^{-2}\right) \tag{3.20}$$

$$X(t) = \frac{1}{\left(1-\gamma^{-2}\right)\left(t_f - t\right) + 1/b} \tag{3.21}$$

$Y(t)$ is the solution of:

$$\dot{Y} = AY + YA^T + DWD^T - YH^T V^{-1} HY \tag{3.22}$$

Where:

$$A = 0,\quad D = 1,\quad H = 1 \tag{3.23}$$

DRE for Y can be written simplicity as:

$$\dot{Y} = W - Y^2 V^{-1} \quad Y(0) = Y_0 \tag{3.24}$$

The steady state value is given by:

$$0 = W - Y^2 V^{-1} \quad \rightarrow Y^2 = WV \tag{3.25}$$

Let:

$$W = 1 \tag{3.33}$$

The RDE is reduced to

$$\dot{Y} = 1 - Y^2 V^{-1} \qquad Y(0) = Y_0 \tag{3.34}$$

Where the solution can be found in [9]:

$$Y(t) = \sqrt{V} + \frac{2\sqrt{V}}{\frac{Y_0 + \sqrt{V}}{Y_0 - \sqrt{V}} e^{\frac{2t}{\sqrt{V}}} - 1} \tag{3.35}$$

One can notice that in the steady state, $Y_{ss} = \sqrt{V}$. This result coincides with the Linear Kalman Filter (LKF) solution. Finally, the gain is given by:

$$\Lambda\left(t, t_f, \gamma^2, V, b, Y_0\right) = \frac{-1}{\left(1-\gamma^{-2}\right)\left(t_f - t\right) + 1/b - \gamma^{-2}\sqrt{V} - \gamma^{-2} \cdot \frac{2\sqrt{V}}{\frac{Y_0 + \sqrt{V}}{Y_0 - \sqrt{V}} e^{\frac{2t}{\sqrt{V}}} - 1}} \tag{3.36}$$

Where:

$$u^* = \Lambda \hat{x} \qquad \hat{x}(t_0) = 0 \tag{3.37}$$

$$\dot{\hat{x}}(t) = \Lambda \hat{x} + Y V^{-1}\left(z - \hat{x}\right) \tag{3.38}$$

For optimality, we demand the following three conditions:

1. Solution to DRE, $Y > 0$, exists $\forall t \in \left[t_0, t_f\right]$.

2. Solution to DRE, $X > 0$, exists $\forall t \in \left[t_0, t_f\right]$.

3. The Spectral Radius Condition (SRC): $1 - \gamma^{-2} YX > 0 \ \ \forall t \in \left[t_0, t_f\right]$.

For the described case, the SBGP, it results in the following inequalities:

$$\left(1-\gamma^{-2}\right)\left(t_f-t\right)+1/b>0 \tag{3.39}$$

$$\sqrt{V}+\frac{2\sqrt{V}}{\frac{Y_0+\sqrt{V}}{Y_0-\sqrt{V}}e^{\frac{2t}{\sqrt{V}}}-1}>0 \tag{3.40}$$

$$\left(1-\gamma^{-2}\right)\left(t_f-t\right)+1/b-\gamma^{-2}\sqrt{V}-\gamma^{-2}\cdot\frac{2\sqrt{V}}{\frac{Y_0+\sqrt{V}}{Y_0-\sqrt{V}}e^{\frac{2t}{\sqrt{V}}}-1}>0 \tag{3.41}$$

The conditions can be summarized by the following expression:

$$\left(1-\gamma^{-2}\right)\left(t_f-t\right)+1/b>\gamma^{-2}\sqrt{V}+\gamma^{-2}\cdot\frac{2\sqrt{V}}{\frac{Y_0+\sqrt{V}}{Y_0-\sqrt{V}}e^{\frac{2t}{\sqrt{V}}}-1}>0 \tag{3.42}$$

From (3.41) we can get the lower bound for γ^2 as follow:

$$\gamma^2>\frac{\sqrt{V}+\left(t_f-t\right)}{1/b+\left(t_f-t\right)}+\frac{2\sqrt{V}}{\frac{Y_0+\sqrt{V}}{Y_0-\sqrt{V}}e^{\frac{2t}{\sqrt{V}}}-1}\cdot\frac{1}{1/b+\left(t_f-t\right)} \tag{3.43}$$

4. Approximation using Steady State LKF (SBGP)

Taking the steady state solution for Y:

$$Y_{ss} = \sqrt{V} \quad (4.1)$$

From that, one can approximate the expression of the gain to be as following:

$$\tilde{\Lambda}\left(t, t_f, \gamma^2, V, b\right) = \frac{-1}{\left(1-\gamma^{-2}\right)\left(t_f - t\right) + 1/b - \gamma^{-2}\sqrt{V}} \quad (4.2)$$

In Figure 2, we run two simulations for the same scenario, one with the approximated gain and one with the exact gain. Simulation parameters: $b = 50, V = 1\left[m^2\right], \gamma^2 = 50, Y_0 = 10\left[m^2\right]$..

One can observe the fast convergence of the approximate to the exact gain.

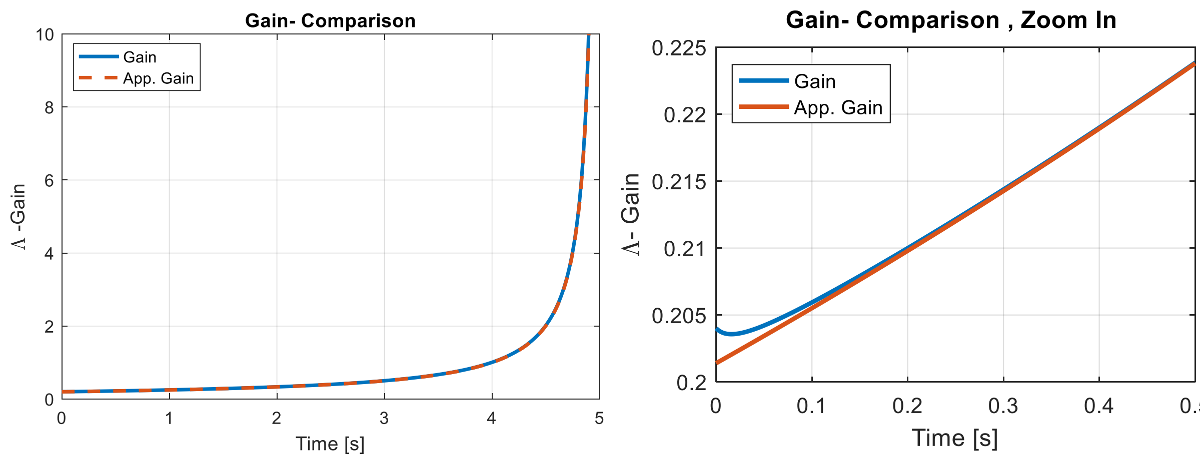

Figure 3- The Convergence in Order to Justify the Approximated Gain

Now, we can investigate the approximate gain expression (4.2).Let γ_c^2 be the minimum value of γ^2 such that the solution to our problem exists, satisfying the three conditions of the last paragraph.

The gain (4.2) must holds $\forall t \in \left[t_0, t_f\right]$. In particular for $t = t_f$. By substituting t_f, the gain approaches to:

$$\tilde{\Lambda}\left(t_f\right) = -\frac{1}{1/b - \gamma^{-2}\sqrt{V}} \quad (4.4)$$

In order to meet the third optimality condition (SRC), the following inequality must hold:

$$1/b - \gamma^{-2}\sqrt{V} > 0 \quad (4.5)$$

Notice that if (4.5) does not hold, we have a finite escape timein (4.3). From the last inequality we can get a lower bound for γ^2 :

$$\gamma^2 > b\sqrt{V} \tag{4.6}$$

We got the critical value for the DA ratio:

$$\left(\gamma_c^{\,2}\right)_1 \triangleq b\sqrt{V} \tag{4.7}$$

From the second inequality we still have:

$$\left(\gamma_c^{\,2}\right)_2 \triangleq 1 \tag{4.8}$$

A generalization of the last two results yields:

$$\gamma_c^{\,2} = \max\left\{b\sqrt{V},1\right\}$$

Under this approximation, let's examine the gain behavior with respect to the noise magnitude. From exploring (4.4), one can notice that as the noise magnitude gets higher, the difference between the positive and negative values in the denominator reduces. As a result, the gain grows. On the other hand, using the (approximate) critical value of the DA ratio, γ_c^2 we get:

$$\tilde{\Lambda} = -\frac{1}{\left(1-1/b\sqrt{V}\right)\left(t_f - t\right)} \tag{4.10}$$

One can notice the opposite behavior of the gain with the noise magnitude: as the noise magnitude gets higher, the gain gets lower. The optimal control and the estimate equation are given by:

$$u^* = \tilde{\Lambda}\hat{x} = -\frac{1}{\left(1-1/b\sqrt{V}\right)\left(t_f - t\right)}\hat{x} \tag{4.11}$$

$$\dot{\hat{x}} = -\left(\left[\left(1-1/b\sqrt{V}\right)\left(t_f - t\right)\right]^{-1} + 1\right)\hat{x} + z \tag{4.12}$$

We may consider the following two limit cases:

$$\left|\Lambda\right|^* \triangleq \lim_{b\to\infty}\left|\Lambda\right| = \lim_{V\to\infty}\left|\Lambda\right| = \frac{1}{\left(t_f - t\right)} \tag{4.13}$$

We get immediately that the last gains are equivalent to the perfect information case (CCG).

5. Numerical Results (SBGP)

The SRC entails:

$$\Omega \triangleq 1-\gamma^{-2}Y(t)\cdot X(t) \tag{5.1}$$

$$\Omega_{\min} = 1-\gamma_c^{-2}Y(t_c)\cdot X(t_c) \tag{5.2}$$

$$\Omega_{\min} = 0 \Rightarrow 1-\gamma_c^{-2}Y(t_c)\cdot X(t_c)=0 \tag{5.3}$$

$$\gamma_c^2 = Y(t_c)\cdot X(t_c) \tag{5.4}$$

In practice, it is not recommended to take the minimum value, and a safety margin from singularity should be considered. We choose $\Omega_{\min} = 0.01$. To demonstrate the results, we run a simulation using a Gaussian White Noise (GWN) with two different values for the standard deviation η. The value of V is adjusted to η by $V=\eta^2$. (This is based on the observation that (3.35) has the form of a standard LKF). The target performs a unit step maneuver: $w=3[m/s]$. The weight of the terminal relative separation (the miss distance) is chosen to be $b=100$.

We examine two cases: the first with a fixed γ and the second with a minimal γ (up to $\Omega_{\min} = 0.01$). The results are given in the following figures.

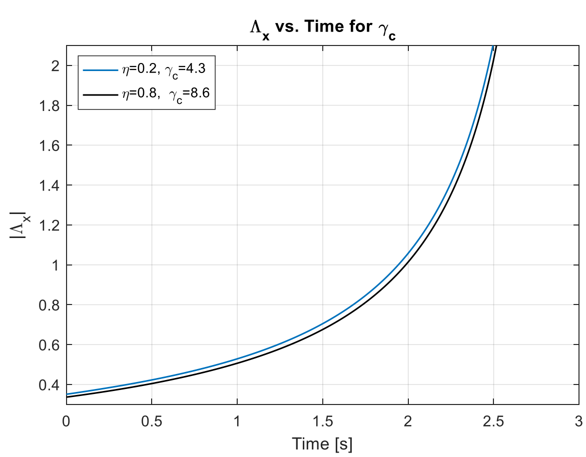

Figure 4- SBGP Gain for Critical DA Ratio

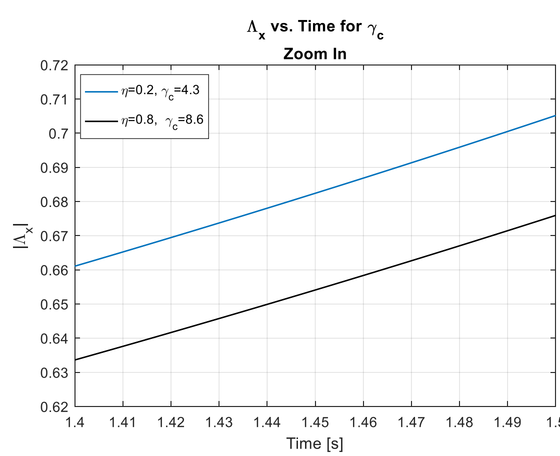

Figure 5- SBGP Gain for Critical DA ratio, Zoom In

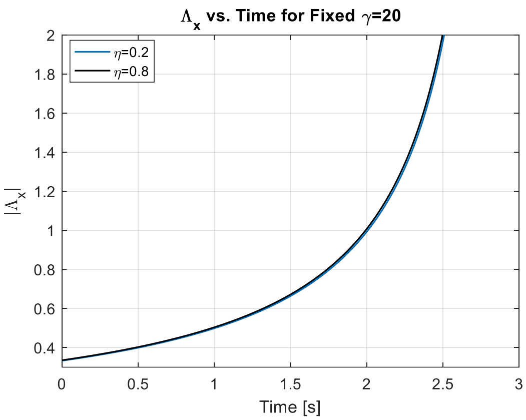

Figure 6- SBGP Gain for Fixed DA Ratio

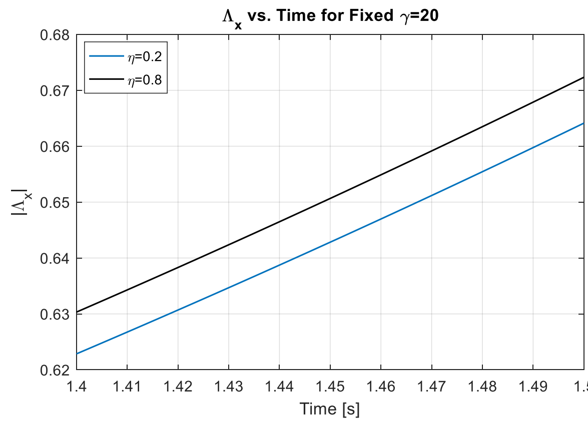

Figure 7- SBGP Gain for Fixed DA Ratio, Zoom In

One can observe that, when the control is designed with a fixed value for γ (the DA ratio), the gain grows as the noise level grows. However, when we consider the near-critical DA value, the gain is reduced as the noise level grows, which is the more intuitive result. Moreover, a comparison between the DA control and the separated control is made (i.e. using the control (3.10) with the estimated x). It appears that by using DA control we may achieve a lower miss-distance (up to 35%), especially with the near-critical DA values.

The simulation was run with the following parameters:

$$\eta = 0.2\left[m^2\right]$$
$$b = 100$$
$$w = 3\left[m/s\right]$$
$$t_f = 3\left[s\right]$$

Monte-Carlo simulation with 1000 runs for the parameters above results in the result in the following table:

Control	$CEP\left[m\right]$	$E\left[u_{Eff}\right]\left[m^2s\right]$
DA, $\gamma = 6$	0.70	48
DA, $\gamma_c = 4.3$	0.57	114
Separated , $\gamma = 6$	0.76	47
Separated , $\gamma_c = 4.3$	0.77	46

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6. Missile Guidance Engagement (MGE)

Assume a linearized model. The Target (T) and the Missile (M) dynamic are approximated with 1st order Autopilot systems. Following the geometry in Figure 8, a set of EOM as state space is presented for the guidance system that described below:

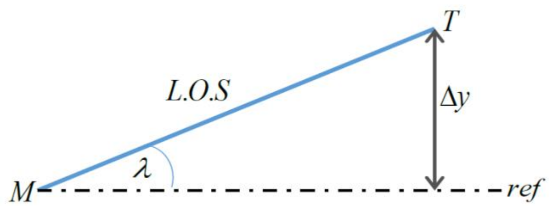

Figure 8 MGE, Problem Geometry

Define: $x_1 \triangleq \Delta y$. Where n_T and a_M are T and M accelerations perpendicular to LOS.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{n}_T \\ \dot{a}_M \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -\theta^{-1} & 0 \\ 0 & 0 & 0 & -T^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ n_T \\ a_M \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ T^{-1} \end{bmatrix} u + \begin{bmatrix} 0 \\ 0 \\ \theta^{-1} \\ 0 \end{bmatrix} w \tag{6.1}$$

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) & n_T(t) & a_M(t) \end{bmatrix}^T \tag{6.2}$$

$$\mathbf{B} = \begin{bmatrix} 0 & 0 & 0 & T^{-1} \end{bmatrix}^T \tag{6.3}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & \theta^{-1} & 0 \end{bmatrix}^T \tag{6.4}$$

$$\mathbf{x}(t), \mathbf{B}, \mathbf{D} \in \mathbb{R}^{4\times1} \tag{6.5}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -\theta^{-1} & 0 \\ 0 & 0 & 0 & -T^{-1} \end{bmatrix} \in \mathbb{R}^{4\times4} \tag{6.6}$$

Where $\mathbf{A}$ (stationary) is the dynamic's matrix. $\mathbf{B}, \mathbf{D}$ are constant control vectors, $\mathbf{x}(t)$ is the states vector.

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{D}\mathbf{w}(t) \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{6.7}$$

Assume imperfect information game. Hence, we don't have full state information. The estimate state vector is given by:

$$\hat{\mathbf{x}}(t) = \begin{bmatrix} \hat{x}_1(t) & \hat{x}_2(t) & \hat{n}_T(t) & \hat{a}_M(t) \end{bmatrix}^T \tag{6.8}$$

The measurement equation:

$$z(t)=H(t)x(t)+v(t) \tag{6.9}$$

For this case, we measure only the LOS angle, λ, such that the observation vector H is given by:

$$H(t)=\left[\left(V_c t_{go}\right)^{-1} \quad 0 \quad 0 \quad 0\right] \tag{6.10}$$

Note that the expression for the LOS rate $\dot{\lambda}$ is derived according to Figure 8 and given by:

$$\lambda=\frac{x_1}{R}=\frac{x_1}{V_c\left(t_f-t\right)}=\frac{x_1}{V_c t_{go}} \tag{6.11}$$

$$\dot{\lambda}=x_1\frac{1}{V_c t_{go}^2}+x_2\frac{1}{V_c t_{go}} \tag{6.12}$$

Considering the following General Performance Measure (GPM) equation:

$$y(t)=C(t)x(t)+F(t)u(t) \tag{6.13}$$

Now, Initial Conditions are unknown. The vectors w,v,x_0 have the following weight matrices:

$$W(t)=W^T(t)>0,\ V(t)=V^T(t)>0,\ Y_0=Y_0^T>0 \tag{6.14}$$

$$J\left(u(\cdot),\tilde{w}(\cdot);t_0,t_f\right)=\left\|y(\cdot)\right\|_2^2-\gamma^2\left\|\tilde{w}(\cdot)\right\|_2^2 \tag{6.15}$$

where $\tilde{w}(t)=\left[w,v,x_0\right]^T$.

For simplicity, Let's assume: $Q\equiv 0$. The cost function explicitly is given by:

$$J=\frac{1}{2}x_f{}^T X_f x_f-\frac{1}{2}\gamma^2 x_0^T Y_0^{-1}x_0+\frac{1}{2}\int_{t_0}^{t_f}\left[u^T u-\gamma^2\left(w^T W^{-1}w+v^T V^{-1}v\right)\right]dt \tag{6.16}$$

Where u is the minimizer and w,v,x_0 are the maximizers.

6.1 Optimal Pursuer (Missile) Strategy:

$$u^*=-B^T X\left(I-\gamma^{-2}YX\right)^{-1}\hat{x} \tag{6.17}$$

$$\Lambda\triangleq-B^T X\left(I-\gamma^{-2}YX\right)^{-1}\triangleq-B^T X\Omega^{-1} \tag{6.18}$$

$$u^*=\Lambda\hat{x} \tag{6.19}$$

Where the estimate equation and the RDE's are given by

$$\dot{\hat{x}}(t) = A\hat{x} + Bu + YH^TV^{-1}(z - H\hat{x}) \tag{6.20}$$

$$-\dot{X} = XA + A^TX - X(BB^T - \gamma^{-2}D^TWD)X \tag{6.21}$$

$$\dot{Y} = AY + YA^T + DWD^T - YH^TV^{-1}HY \qquad ,Y(t_0) = Y_0 \tag{6.22}$$

7. Numerical Results (MGE)

Assume the following parameters:

$$\begin{aligned} X_f &= diag\{b,0,0,0\} \\ V_c &= 300[m/s] \\ x_0 &= 0, Y_0 = I_4, W = 3, b = 1000 \end{aligned} \tag{7.1}$$

We explore 3 cases for the MGE. Firstly we introduce the behavior of the Missile gain with fixed and critical DA ratio. Secondly, we introduce a comparison between the DA control, the Separation Control, the perfect state control and a well-known Guidance control -the PN. Lastly, we introduce the sensitivity of the control to flight- time by comparing the resultant miss-distance for the DA and the Separation control.

7.1. DA Ratio (Fixed and Critical):

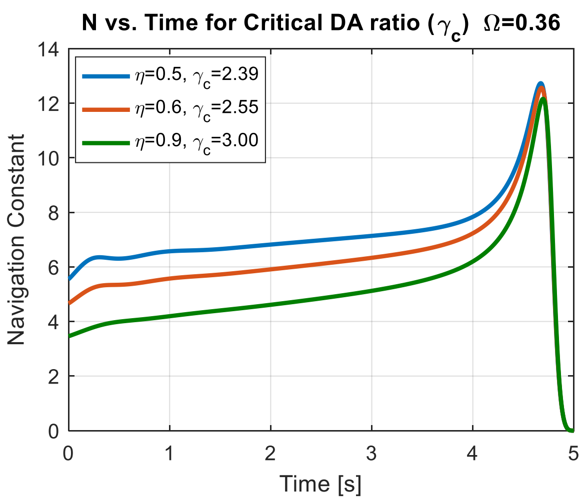

Figure 9- MGE, N Constant for Critical DA Ratio

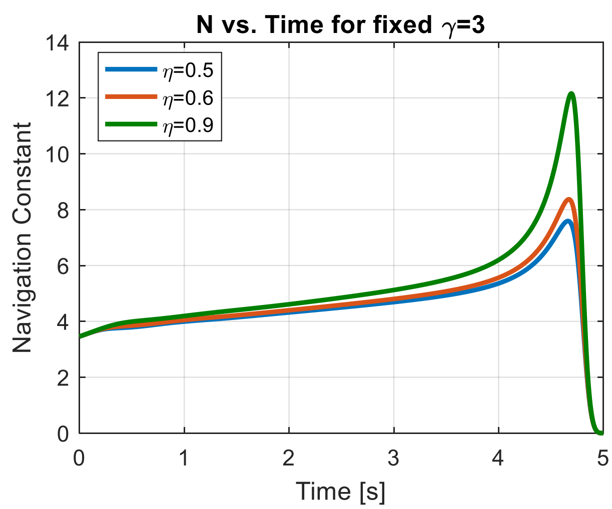

Figure 10- MGE, N Constant for Fixed DA Ratio

7.2. Comparison between 4 Controls:

Disturbances Attenuation Control: $u_1 = -B^T X\left(I - \gamma^{-2} YX\right)^{-1} \hat{x}$

Perfect State Control: $u_2 = -B^T Xx$

Separation Control: $u_3 = -B^T X\hat{x}$

Proportional Navigation Control: $u_4 = -\frac{3}{t_{go}^2}\left(\hat{x}_1 + t_{go}\hat{x}_2\right)$

$$u_i = \begin{cases} u_i & |u_i| < u_{sat} \\ u_{sat} \cdot \mathrm{sgn}(u_i) & |u_i| > u_{sat} \end{cases} \qquad i \in \{1..4\}$$

In order to examine the performance of the control we take the previous scenario and simulate again the engagements with two additional controls: the Separation Control and the Perfect State Control. Noise Level is chosen to be $50[\mu rad]$ and $\gamma = 2.5$. Comparison is given below.

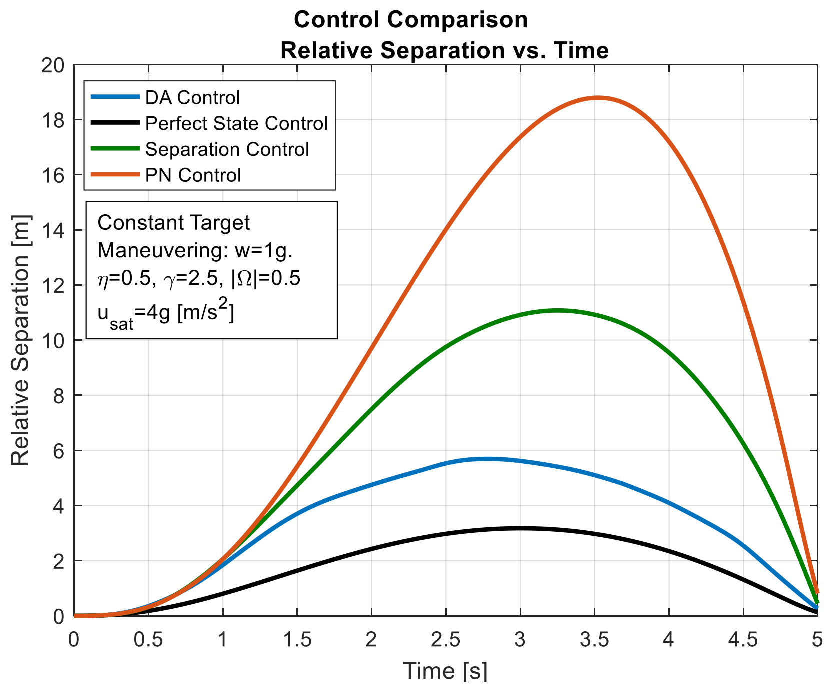

Figure 11-MGE Control Comparison

Control	CEP $[cm]$	$E[u_{Eff}]\,[m^2/s^3]$
$u_1 = -B^T S\Omega^{-1}\hat{x}$	30	392
$u_2 = -B^T Sx$	14	284
$u_3 = -B^T S\hat{x}$	52	532
$u_4 = -\frac{3}{t_{go}^2}\left(\hat{x}_1 + t_{go}\hat{x}_2\right)$	128	960

7.3. Robustness to Flight Time

Consider that engagement duration isn't fixed. In this case the influence of the Noise Level for some value of t_f were checked. Additionally, comparison between the Separation control and the DA control was done.

$$t_f \in [3,15][\mathrm{s}]$$

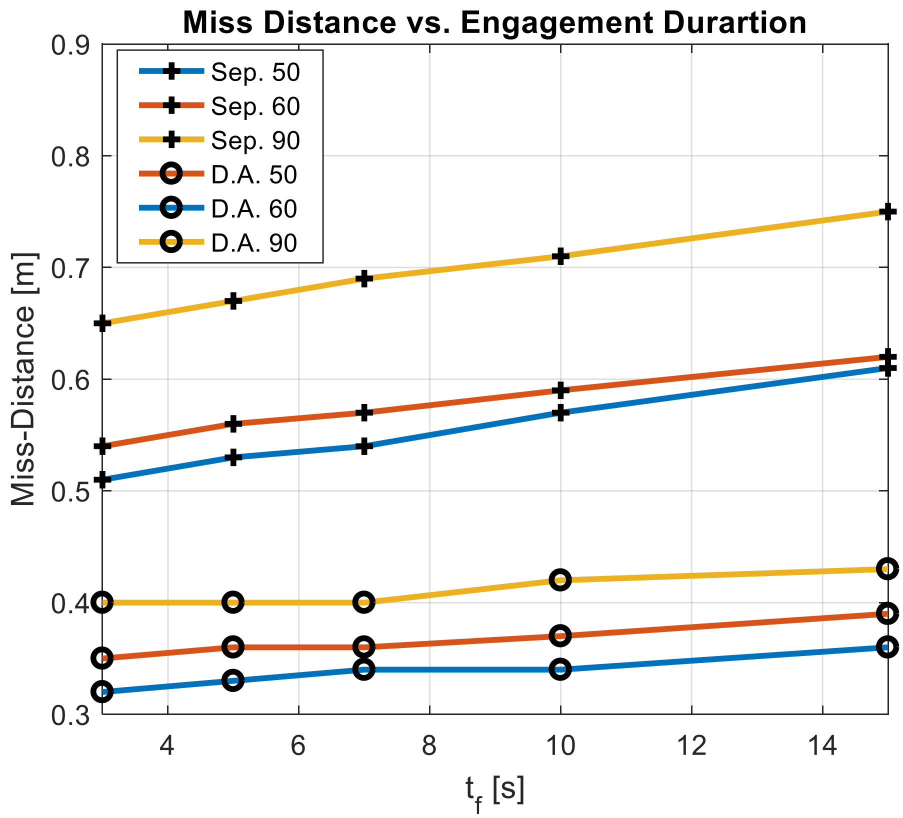

Figure 12 MGE, Miss Distance as a Function of Engagement Duration

One can notice that the DA control is much robust to change in t_f than the Separation control. Moreover, the miss-distance of the former is much higher.

Pursuit Evasion Games with Imperfect Information Revisited
Barak Or, Joseph Z. Ben-Asher, Isaac Yaesh

8. Summary

The problem of DA with imperfect information pattern has been revisited. First the equivalence between the two main DA control solution formulations was established for the finite-time horizon case. Then, two representative cases were introduced, with perfect and imperfect information. The detailed analysis of these two DA problems revealed the advantage of using the critical value of DA ratio over a fixed DA ratio. Although we focused in simple guidance problems, these problems seem to capture some of the main characteristics of problems of higher complexity.

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Appendix A

DA ratio as a function of time:

$$\left(1-\gamma^{-2}\right)\left(t_f-t\right)+1/b-\gamma^{-2}\sqrt{V}>0 \qquad (a.1)$$

$$\left(t_f-t\right)-\gamma^{-2}\left(t_f-t\right)+1/b-\gamma^{-2}\sqrt{V}>0 \qquad (a.2)$$

$$\gamma^2\left(t_f-t\right)-\left(t_f-t\right)+\gamma^2/b-\sqrt{V}>0 \qquad (a.3)$$

$$\gamma^2\left(t_f-t+1/b\right)>\left(t_f-t\right)+\sqrt{V} \qquad (a.4)$$

$$t_{go}\triangleq t_f-t \qquad (a.5)$$

$$\gamma^2\left(t\right)>\frac{t_{go}+\sqrt{V}}{t_{go}+1/b}\triangleq\psi\left(t_{go}\right) \qquad (a.6)$$

$$\lim_{t_{go}\to\infty}\psi\left(t_{go}\right)=\lim_{t_{go}\to\infty}\frac{1+\sqrt{V}/t_{go}}{1+1/\left(b\cdot t_{go}\right)}=1 \qquad (a.7)$$

$$\lim_{t_{go}\to 0}\psi\left(t_{go}\right)=\lim_{t_{go}\to 0}\frac{t_{go}+\sqrt{V}}{t_{go}+1/b}=b\sqrt{V} \qquad (a.8)$$

$$\gamma_c^{\,2}=\max\left\{b\sqrt{V},1\right\} \qquad (a.9)$$